Introduction to diffraction theory → Fourier Optics

**Refraction:** Snell's Law  \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \)

Rays "bend" on boundary due to boundary conditions prevailing phase.

**Diffraction:** defined by Sommerfeld:
"any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction"

- Diffraction occurs due to confinement to sizes comparable to 1-wavelength in lateral (transverse) direction.

**History:**
- 1665 - first accurate report
- Grimaldi using corpuscular theory of light - particles predicted "sharp" shadow lines!!!
- Grimaldi observed deviations in experiments → light in regions of predicted "shadow" did not work
- 1678 Christian Huygens explained intuitively using secondary spherical sources
1804 Thomas Young introduces effect of interference on screen.

1818 Fresnel brings Huygens & Young ideas together and establishes diffraction theory by allowing fields from secondary sources to interfere. Achieves excellent accuracy in calculation of fields & correl. with experiment.

1860 Maxwell identifies "light" as electromagnetic field/wave.

1882 Kirchhoff puts ideas of Huygens & Fresnel on firm theoretical grounds.

1892 Controversy with Kirchhoff assumptions by Poincare & Sommerfeld concluding Kirchhoff formulation is first-order approximation.

Sommerfeld modified Kirchhoff theory and realized some of the physical assumptions concerns.

We will follow this diffraction theory formulation but first, some mathematical preliminaries and definitions.
Let light disturbance be given in point $P$ in space at time $t$ be scalar value (usually vectorial): $u(P, t)$. We will deal with monochromatic fields:

$$u(P, t) = A(P) \cos(2\pietail t - \phi(P))$$

$A(P)$ & $\phi(P)$ are amplitude and phase of the fields in $P$, $\omega$-opt. freq.

$$\omega = \frac{c}{\lambda_0}$$

Compact notation via complex signal:

$$u(P, t) = \text{Re} \left\{ U(P) \exp(-j2\pietail t) \right\}$$

where $U(P)$ - called phasor

$$U(P) = A(P) \exp[j\phi(P)]$$

The disturbance "must" satisfy Maxwell eqs or scalar wave eq:

$$\nabla^2 u - \frac{n^2}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

$$D(t) = \frac{2}{c^2} \frac{2\pi n}{\lambda_0} \lambda_0 \kappa \lambda_0$$

For complex ampl. $U$

$$(\nabla^2 + k^2)U = 0$$

Helmholtz eq.

Intensity: Photodetectors do not respond to optical carrier, they detect power flow: $H$. direction of $k$
Let $\vec{B} = \text{Re} \{ E \exp[-i(kr + \omega t)] \}$ be a plane wave and $I$ is

$$P = \frac{E \cdot E^*}{2\eta}$$

where $E$ is amplitude of the electric field vector and $\eta = \sqrt{\mu / \varepsilon}$ characteristic impedance (copper, $\eta = 377 \Omega$).

Total power incident on a surface of area $S$ is

$$P = \iiint \frac{k \cdot \hat{n}}{|k|} \, d\mathbf{x} \, d\mathbf{y}$$

$\hat{n}$ is surf. normal

$k / |k|$ a unit vector

$k \cdot \hat{n} \rightarrow k$ encounters for the larger surface (for short)

When $k \cdot \hat{n}$

$$P \cdot S = P \leftarrow \text{power detected}$$

Back to our case, we define intensity

$$I = |u(P)|^2$$

Intensity is a measurable attribute of optical field. $[I] = \text{W/cm}^2$ or $\text{W/m}^2$.

Instantaneous intensity $I(P,t) = |u(P,t)|^2$.

Intensity of diffracted pattern will be measurable!
The Huygens-Fresnel Principle

- Diffraction aperture is illuminated in positive \( z \).
- Calculate the diffracted field in \((x, y)\) plane parallel to \((\xi, \eta)\) plane.

Reproducing the H-F principle:

\[
U(P_0) = \frac{1}{j\lambda} \iint_U U(P_i) \frac{\exp(jkr_{01})}{r_{01}} \cos \Theta \, ds
\]

\( \Theta \) - angle between \( (\hat{h}, \vec{r}_{01}) \); \( \cos \Theta = \frac{z}{r_{01}} \)

yielding

\[
U(P_0) = \frac{2z}{j\lambda} \iint U(P_i) \frac{\exp(jkr_{01})}{r_{01}^2} \, ds
\]

and

\[
r_{01} = \sqrt{(x-x')^2 + (y-y')^2 + z^2}
\]

So far we used 2 approximations:
- scalar fields
- \( r_{01} \gg \lambda \) observation not in the near
The Fresnel Approximation: 

we use binomial expansion (Taylor) of $\sqrt{1 + \theta^2}$

$\sqrt{1 + \theta^2} = 1 + \frac{1}{2} \theta^2 - \frac{1}{8} \theta^4 + \ldots$

Next we rewrite

$r_{01} = 2 \sqrt{1 + \left(\frac{x - \xi}{\tau}\right)^2 + \left(\frac{y - \eta}{\tau}\right)^2}$

or

$r \approx 2 \left[ 1 + \frac{1}{2} \left(\frac{x - \xi}{\tau}\right)^2 + \frac{1}{2} \left(\frac{y - \eta}{\tau}\right)^2 \right]^2$

Next question, should we keep all terms in $r \approx$?

- If $(\xi,\eta)$ & $(x,y)$ apertures extend
  is $r \ll \tau$ → we can replace
  $r_{01}$ in the denominator by $\tau$

- In phase it is more tricky!
  - multiply by $k$ which is large @ freq.
  - phase varies on scale of $\lambda$
    and our variations in $r_{01}$ are
    many $\lambda$ even when transverse
    size is $r \ll \tau$.

So we get:

$\mathcal{U}(x,y) = \frac{\exp(i k \tau)}{j \lambda^2} \int \int \mathcal{U}(\xi,\eta) \exp \left[ \frac{j k}{2 \tau} (x - \xi)^2 + (y - \eta)^2 \right] d\xi d\eta$

We incorporated aperture $\Sigma$ onto $\mathcal{U}(\xi,\eta)$. 
Last eq. in engineering → convolution
we can rewrite:
\[ U(x,y) = \int \int U(x',y') h(x-x', y-y') \, dx' \, dy' \]

with convolution kernel
\[ h(x,y) = \frac{e^{ikz}}{j\lambda z} \exp\left[ \frac{ik}{2z} (x^2 + y^2) \right] \]

Another way to write the Fresnel Approximation:
\[ U(x,y) = \frac{e^{ikz}}{j\lambda z} \frac{e^{ik/2} (x^2 + y^2)}{2\pi} \int \int U(x',y') \, e^{ik/2 (x'^2 + y'^2)} \, dx' \, dy' \]

so Fresnel diffraction \( \propto \) FT of field in aperture \( \times \) quadratic phase.

Note: \( e^{jkr} \) - diverging spherical wave
\( e^{-jkr} \) - converging

Similarly \( e^{\pm jk/2 (x^2 + y^2)} \)

Accuracy of Fresnel Approximation.

Need to consider effect of term in \( V1+b^2 \) that was ignored \( \frac{1}{8} b^2 \) and make it smaller than say 1 radian → then Fresnel approximation is valid, i.e. \( \frac{b^2}{8} \ll 1 \)
or \( z^3 \gg \frac{1}{4 \lambda} \left[ (x-x)^2 + (y-y)^2 \right]_{\text{max}} \)
Example: \( \lambda = 0.5 \text{\( \mu \text{m} \)}, \ \Phi = \Phi_y = 1 \text{\( \text{cm} \)} \)

\[ z \gg 25 \text{\( \text{cm} \)} \]

This was a stringent condition. We can make it less stringent (see book).

**Fresnel Approximation:**

We can also find distance that phase in Fresnel integral with FT formulation can be dropped. This occurs when

\[ z \gg \frac{2}{k(\Phi^2 + \gamma^2)_{\text{max}}} \]

This is called the Fresnel approximation yielding

\[ U(x, y) = \exp[ikz] \exp\left[\frac{ik}{2z}(x^2 + y^2)\right] \int [U(\xi, \eta)] \exp\left[-\frac{ik}{2z}(\xi^2 + \eta^2)\right] d\eta d\xi \]

**Definition of spatial frequency**

\[ f_x = \frac{x}{kz}; \quad f_y = \frac{y}{kz} \]

and

\[ U(f_x, f_y) \approx \mathcal{F}\{U(x, y)\} \]

**Validity:**

\[ \lambda = 0.6 \text{\( \mu \text{m} \)}; \ \Phi = 2.5 \text{\( \text{cm} \)}; \quad z \gg 1.6 \text{\( \text{km} \)} \]

less stringent "antenna designer" formula \( z \gg \frac{2D^2}{\lambda} \)
Fourier Transform:

\[ \mathcal{F}\{g(x,y)\} = \mathcal{G}(f_x, f_y) = \iint g(x,y) \exp[-i 2\pi (f_x x + f_y y)] \, dx \, dy \]

\[ \mathcal{F}^{-1}\{ \mathcal{G}(f_x, f_y) \} = g(x,y) = \iint \mathcal{G}(f_x, f_y) \exp[i 2\pi (f_x x + f_y y)] \, df_x \, df_y \]

\( \delta \)-function

\[ \delta(x,y) = \lim_{N \to \infty} N^2 \exp[-N^2 \pi (x^2 + y^2)] \]

\[ \mathcal{F}\{ N^2 \exp[-N^2 \pi (x^2 + y^2)] \} = \exp\left[-\frac{\pi (f_x^2 + f_y^2)}{N^2}\right] \]

Thus

\[ \mathcal{F}\{ \delta(x,y) \} = \lim_{N \to \infty} \{ \exp\left[-\frac{\pi (f_x^2 + f_y^2)}{N^2}\right] \} = 1 \]

Properties:

- Linearity \[ \mathcal{F}\{ \alpha g + \beta h \} = \alpha \mathcal{F}\{g\} + \beta \mathcal{F}\{h\} \]
- Similarity (scaling) \[ \mathcal{F}\{g(ax, by)\} = \frac{1}{|ab|} \mathcal{G}\left(\frac{f_x}{a}, \frac{f_y}{b}\right) \]
- Shift \[ \mathcal{F}\{g(x-a, y-b)\} = \mathcal{G}(f_x, f_y) \exp[-i 2\pi (f_x a + f_y b)] \]
- Parseval's \[ \iint |g(x,y)|^2 \, dx \, dy = \iint |\mathcal{G}(f_x, f_y)|^2 \, df_x \, df_y \]
- Convolution: \[ \mathcal{F}\{g(x,y) * h(x,y)\} = \mathcal{F}\{g\} \cdot \mathcal{F}\{h\} \]
- A correlation

\[ \int \int g(x,y) f^*(x-x', y-y') \, dx \, dy = |G|^2 \]

Examples: From the diffraction pattern.

A. Rect aperture

\[ t_A(x,y) = \text{rect} \left( \frac{x}{2w_x} \right) \text{rect} \left( \frac{y}{2w_y} \right) \]

\[ U(x,y) = \frac{e^{i k z}}{j x z} \sum_{\ell_x, \ell_y} U(\ell_x, \ell_y) \]

\[ = \frac{1}{j 2\pi f_x} \frac{1}{j 2\pi f_y} \exp \left( -j 2\pi f_x f_x \right) \exp \left( -j 2\pi f_y f_y \right) \]

\[ = \frac{1}{2\pi f_x} \frac{1}{2\pi f_y} \sin \left( 2\pi f_x f_x \right) \sin \left( 2\pi f_y f_y \right) \]

\[ \delta \left( f_x, f_y \right) = \text{Area of aperture} \]

\[ I = \int \int |U(x,y)|^2 = \frac{A^2}{12} \cdot \sin^2 \left( \frac{2\pi f_x f_x}{12} \right) \sin^2 \left( \frac{2\pi f_y f_y}{12} \right) \]
\[ \text{Sinc}(x) = \frac{\sin \pi x}{\pi x} \]

\[ x = 0 \quad \text{Sinc}(x) = 1 \]

\[ \pi x = \pi \quad \text{Sinc}(x) = 0 \]

\[ \text{or} \quad x = \frac{\lambda}{2W^2} \]

\[ \frac{x_1 \cdot 2W}{\lambda} = 1 \quad ; \quad x_1 = \frac{\lambda}{2W^2} \quad \text{or} \]

The width of main lobe is \( 2 \cdot x_1 \cdot W \)

Circular aperture:

\[ \Delta x = \frac{\lambda}{2W^2} \]

\[ \pm A(\theta) = \text{circ} \left( \frac{\theta}{w} \right) = \text{circ} \left( \frac{\phi}{W} \right) \]

This is circularly symmetric function.

We need find how to deal with such functions in optics → Fourier-Bessel Transforms.
let \( g(r, \Theta) = g_r(r) \)

Then we make coord. transf. from \((x, y)\) to polar:

\[
\begin{align*}
   r &= \sqrt{x^2 + y^2} \\
   \Theta &= \tan^{-1} \frac{y}{x} \\
   x &= r \cos \Theta \\
   y &= r \sin \Theta \\
\end{align*}
\]

\[
\begin{align*}
   F_x &= \rho \cos \phi \\
   F_y &= \rho \sin \phi
\end{align*}
\]

\[
F \{ g \} = G_0 (\rho, \phi); \text{ applying coord. transf. to } FT \text{ in } (x, y) \text{ we get:}
\]

\[
G_0 (\rho, \phi) = \int_0^{2\pi} d\Theta \int_0^\infty dr \, g_r(r) \exp[-j2\pi \rho (\cos \Theta \cos \phi + \sin \Theta \sin \phi)]
\]

or combining \( \cos \Theta \cos \phi + \sin \Theta \sin \phi \)

\[
\begin{align*}
   &= \int_0^{2\pi} d\Theta \int_0^\infty dr \, g_r(r) \exp[-j2\pi \rho r \cos(\Theta - \phi)] \\
   &= \int_0^\infty dr \, g_r(r) \int_0^{2\pi} d\Theta \exp[-j2\pi \rho r \cos(\Theta - \phi)]
\end{align*}
\]

\[
Bessel \ function
\]

\[
J_0 (a) \equiv \frac{1}{2\pi} \int_0^{2\pi} \exp[-ja \cos(\Theta - \phi)] d\Theta
\]

Bessel function of 1st kind, 0-order
Next: 

\[ a = 2\pi r_p \quad \text{and} \quad G_0(p, \phi) = G_0(\theta) = 2\pi \int_0^\infty r g(r) J_0(2\pi r_p) \, dr \]

Now we substitute our circular aperture,

\[ \phi \]

\[ 2\pi \int_0^\infty r J_0(2\pi r_p) \, dr = G_0(\theta) \]

\[ \text{change var:} \quad 2\pi r g = \xi \]

\[ 2\pi dr = d\xi \]

\[ r = w \Rightarrow 2\pi pw \]

\[ \frac{1}{(2\pi p)^2} \int_0^\infty \xi J_0(\xi) \, d\xi = \frac{1}{(2\pi p)^2} \frac{d}{dx} [xJ_1(x)] \]

Results in:

\[ \frac{1}{(2\pi p)^2} 2\pi \left[ \frac{\xi}{2} J_1(\xi) \right]_0^{2\pi pw} = \frac{1}{(2\pi p)^2} \frac{d}{dx} [xJ_1(x)] \]

\[ = \frac{w}{p} \cdot J_1(2\pi pw) = \pi w^2 \cdot \frac{J_1(2\pi pw)}{\pi w^2} \]

\[ A \equiv \pi w^2 \]

\[ G_0(\theta) = \frac{A \cdot J_1(2\pi pw)}{\pi pw} \]

or we can convert to "meters";

\[ G_0(\frac{r'}{\lambda^2}) = A \cdot \frac{J_0(\frac{kr'}{\lambda^2})}{kr'w/2} \]
Finally, we substitute into Fraunhofer diffraction:

\[ U(r') = e^{i k z} \frac{e^{i k r'^2}}{i k r'} A \left[ 2 J_1(k r' w/2) \right] \]

Intensity:

\[ I(r') = \left( \frac{A}{\lambda} \right)^2 \left[ 2 J_1(k r' w/2) \right]^2 \]

This is referred to as the Airy pattern. Table values can be found in books.

\[ \frac{2 \pi r'}{\lambda} = 1.22 \]

Example: Annular Aperture

\[ I_f \propto \left[ \frac{J_1(k r' w/2)}{kr' w/2} - \frac{J_1(k r' e w/2)}{kr' e w/2} \right]^2 \]

\[ \epsilon = \frac{1}{2} \]
Annular aperture increases the resolution but decreases the contrast.

In the limit: when \( \varepsilon \to 1 \)

\[ f(r) \approx \delta(r-w) \text{ and} \]

\[ F(r') \propto G_0(p) = 2\pi \int_0^\infty r \delta(r-w) J_0(2\pi rp) \, dr \]

\[ I_{F(r')} \propto \left| J_0(kr'w/2) \right|^2 \]
Example: Thin sinusoidal Amplitude grating

\[ t_A(x,y) = \left[ \frac{1}{2} + \frac{m}{2} \cos 2\pi f_0 x \right] \text{rect} \frac{y}{2\omega} \text{rect} \frac{\omega}{2\omega} \]

where

\[ t_A(x,y) \]

Use convolution theorem:

\[ \mathcal{F} \left\{ \frac{1}{2} + \frac{m}{2} \cos 2\pi f_0 x \right\} = \frac{1}{2} \delta(f_x, f_y) + \]

\[ + \frac{m}{4} \delta(f_x - f_0, f_y) + \frac{m}{4} \delta(f_x + f_0, f_y) \]

and

\[ \mathcal{F} \left\{ \text{rect} \frac{y}{2\omega} \text{rect} \frac{\omega}{2\omega} \right\} = A \frac{\text{sinc}(2\omega f_x)}{2\omega} \frac{\text{sinc}(2\omega f_y)}{2\omega} \]

Next

\[ U(x,y) = \frac{A}{(2\pi)^2} e^{-\frac{j \lambda_0}{2} (x^2 + y^2)} \text{sinc} \left[ \frac{2\omega y}{\lambda_0} \right] \]

\[ \left[ \text{sinc} \left( \frac{2\omega(x)}{\lambda_0^2} \right) + \frac{m}{2} \text{sinc} \left[ \frac{2\omega}{\lambda_0^2} (x + f_0 \lambda_0) \right] + \frac{m}{2} \text{sinc} \left[ \frac{2\omega}{\lambda_0^2} (x - f_0 \lambda_0) \right] \right] \]

Note: if many periods \( \to f_0 \gg \frac{1}{\omega} \)

Then we will have negligible overlap...
Ignore cross terms in intensity calculation.

\[ I(x,y) \sim \left( \frac{A}{2a^2} \right)^2 \sin^2 \left( \frac{2\pi y}{\Lambda} \right) \left\{ \sin^2 \frac{2\pi x}{\Lambda} + \frac{m^2}{4} \sin^2 \left[ \frac{2\pi}{\Lambda} (x \pm f_0 z) \right] + \frac{m^2}{4} \sin^2 \left[ \frac{2\pi}{\Lambda} (x \mp f_0 z) \right] \right\} \]

Note from \( \frac{m^2}{4} \left\{ \frac{1}{2} + \frac{m}{4} \cos \ldots \right\} = \frac{1}{2} \delta \ldots + \frac{m}{4} \ldots + \ldots + \frac{m}{4} \ldots \)

we observe that efficiency is

in \( \pm \) th & \( \pm \) 1st orders is

\[ \eta_0 = \frac{1}{4} \]

\[ \eta_{\pm 1} = \frac{m^2}{16} \]

\[ \eta_{\mp 1} = \frac{m^2}{16} \]

even \( \eta \) at \( m = 1 \)

the power in \( \pm \) orders

is at most \( \frac{1}{16} = 6.25\% \)

The total efficiency is

\[ \frac{1}{4} + \frac{m^2}{8} \rightarrow \text{expected when encountering absorption in the grating!} \]
Plane wave spectrum decomposition

Let

\[ u(x, y; z=0) \text{ is known} \]

We can find its spectrum as:

\[ A_0(f_x, f_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y; 0) e^{-j2\pi(f_x x + f_y y)} \, dx \, dy \]

with inverse

\[ u(x, y; 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_0(f_x, f_y) e^{j2\pi(f_x x + f_y y)} \, df_x \, df_y \]

Next go distance \( z \) to observe plane

\[ u(x, y; z) \] with its

\[ A(f_x, f_y; z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y; z) e^{-j2\pi(f_x x + f_y y)} \, dx \, dy \]

and

\[ u(x, y; z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(f_x, f_y; z) e^{j2\pi(f_x x + f_y y)} \, df_x \, df_y \]

\( u(x, y; z) \) must satisfy Helmholtz eq.

\[ (\nabla^2 + k^2) u(x, y; z) = 0 \]

We substitute Helmholtz eq.

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{d^2}{dx^2} A(f_x, f_y; z) \right] e^{-j2\pi(f_x x + f_y y)} \, df_x \, df_y + \]

\[ + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(f_x, f_y; z) \left[ (j2\pi f_x)^2 + (j2\pi f_y)^2 \right] e^{-j2\pi(f_x x + f_y y)} \, df_x \, df_y \]
\[ + k^2 \iint A(f_x, f_y; z) e^{j 2\pi (f_x x + f_y y)} \, df_x \, df_y = 0 \]

\[ \frac{\partial^2 A(x, y; z)}{\partial z^2} + \left\{ - \left(\frac{2\pi}{\lambda} x \right)^2 + \left(\frac{2\pi}{\lambda} y \right)^2 \right\} + k^2 A(x, y; z) \, df_x \, df_y = 0 \]

and

\[ \frac{d^2 A(x, y; z)}{dz^2} + \left(\frac{2\pi}{\lambda} \right)^2 \left[ 1 - (f_x)^2 - (f_y)^2 \right] A(x, y; z) = 0 \]

Solving it we get

\[ A(f_x, f_y; z) = A' \exp \left\{ j \frac{2\pi}{\lambda} \sqrt{1 - (f_x)^2 - (f_y)^2} \right\} \]

and \( z = 0 \), \( A(f_x, f_y; 0) = A' \)

so we get

\[ A(f_x, f_y; z) = A(f_x, f_y; 0) \exp \left\{ j \frac{2\pi}{\lambda} \sqrt{1 - (f_x)^2 - (f_y)^2} \right\} \]

\[ u(x, y; z) = \mathcal{F}^{-1} \left\{ A(f_x, f_y; z) \right\} = \mathcal{F}^{-1} \frac{u(x, y; 0)}{A(f_x, f_y; 0)} + \mathcal{F}^{-1} \left\{ j \frac{2\pi}{\lambda} \sqrt{1 - (f_x)^2 - (f_y)^2} \right\} \]

Impulse response of a filter

The filter transfer function is

\[ H(f_x, f_y) = \frac{A(f_x, f_y; z)}{A(f_x, f_y; 0)} \]
Example: Fresnel Approximation

We need an expression of a convolution

\[ h(x, y) = \int \int U(\xi, \eta) h(x-\xi, y-\eta) \, d\xi \, d\eta \]

with kernel

\[ h(x, y) = \frac{e^{jkz}}{jz^2} \exp \left[ \frac{jk}{2z} (x^2 + y^2) \right] \]

Take FT of \( h \)

\[ \mathcal{F}\{h(x, y)\} = H(f_x, f_y) = e^{jkz} \exp \left[ -jkz \left( \frac{f_x^2 + f_y^2}{2} \right) \right] \]

in comparison, take one accurate

\[ H(f_x, f_y) \]

and apply binomial expansion

\[ \sqrt{1 - \left(\frac{f_x}{2}\right)^2 - \left(\frac{f_y}{2}\right)^2} \approx 1 - \frac{(f_x)^2}{2} - \frac{(f_y)^2}{2} \]

we get same result!!!
Talbot Images.

Consider grating
\[ L_A (x,l) = \frac{1}{2} \left( 1 + m \cos \left( \frac{2\pi x}{L} \right) \right) \]

Calculation in Fresnel approx:
\[ H(f_x, f_y) = \exp \left[ -j \frac{\pi \lambda z}{2} \left( f_x^2 + f_y^2 \right) \right] \]
(omitted const)

\[ \mathcal{F}\{L_A^2\} = \frac{1}{2} \delta(f_x, f_y) + \frac{m}{4} \delta(f_x - \frac{1}{L}, f_y) + \frac{m}{4} \delta(f_x + \frac{1}{L}, f_y) \]

Now we multiply \( H \) by \( \mathcal{F}\{L_A^2\} \)

\( H(0,0) = 1, \quad H(\pm \frac{1}{L}, 0) = \exp \left[ -j \frac{\pi \lambda z}{L^2} \right] \)

After propagating distance \( z \) behind grating
\[ \mathcal{F}\{U(x, y)\} = \frac{1}{2} \delta(f_x, f_y) + \frac{m}{4} e^{-j \frac{\pi \lambda z}{L^2}} \delta(f_x - \frac{1}{L}, f_y) + \frac{m}{4} e^{-j \frac{\pi \lambda z}{L^2}} \delta(f_x + \frac{1}{L}, f_y) \]
Inverse:

\[ U(x,y) = \frac{1}{2} + \frac{m}{4} e^{-\frac{\pi^2 I_0^2}{L^2}} e^{j \frac{2\pi x}{L}} + \frac{m}{4} e^{\frac{\pi^2 I_0^2}{L^2}} e^{j \frac{2\pi x}{L}} \]

with simplification:

\[ U(x,y) = \frac{1}{2} \left[ 1 + m e^{j \frac{\pi I_0^2}{L^2}} \cos \frac{2\pi x}{L} \right] \]

& Intensity

\[ I(x,y) = \frac{1}{4} \left[ 1 + 2 \cos \left( \frac{\pi I_0^2}{L^2} \right) \cos \frac{2\pi x}{L} + \right. \]

\[ \left. + m^2 \cos^2 \left( \frac{2\pi x}{L} \right) \right] \]

Consider 3 cases:

1) Let distance \( z \) satisfy

\[ \frac{\pi I_0^2}{L^2} = 2n \pi \quad \text{or} \quad z = \frac{2n L^2}{\lambda} \]

\( n \)-integer, then intensity is

\[ I = \frac{1}{4} \left[ 1 + m \cos \frac{2\pi x}{L} \right]^2 \rightarrow \text{perfect image of the grating} \]

No lens! called Talbot images

Physics: spectral content

0, 1\text{st}, 2\text{nd}\text{ orders are supported from periodic images!}
2. Suppose

\[ \frac{\pi \lambda^2}{L^2} = (2n+1) \frac{\pi}{\lambda} \quad \text{odd} \quad \pi \]

or \( \pi = \frac{(2n+1)\lambda^2}{L^2} \)

Then \( \cos \left( \frac{\pi \lambda^2}{L^2} \right) = 0 \) and

\[ I(x,y) = \frac{1}{4} \left[ 1 + m^2 \cos^2 \left( \frac{2\pi x}{L} \right) \right] = \]

\[ = \frac{1}{4} \left[ (1 + m^2) + \frac{m^2}{2} \cos \left( \frac{4\pi x}{L} \right) \right] \]

with 180° spatial phase shift in intensity "contrast reversal"

3. Consider

\[ \frac{\pi \lambda^2}{L^2} = (2n-1) \frac{\pi}{\lambda} \quad \text{or} \quad \pi = \frac{(2n-1)\lambda^2}{L^2} \]

Then \( \cos \left( \frac{\pi \lambda^2}{L^2} \right) = 0 \) and

\[ I(x,y) = \frac{1}{4} \left[ 1 + m^2 \cos^2 \left( \frac{2\pi x}{L} \right) \right] = \]

\[ = \frac{1}{4} \left[ (1 + m^2) + \frac{m^2}{2} \cos \left( \frac{4\pi x}{L} \right) \right] \]

Twice frequency of original called Talbot subimage
grating

Talbot

Phase reversed

Talbot image

Phase reversed

Sub images
Wave Optics Analysis of Coherent Optical Systems

Lenses - are important devices. Hubble telescope! → software is good, but only in special cases.

Approach: Wave optics in imaging results - consistent with geometric optics.

A thin lens as a phase transformer.

\[
\Phi(x, y) = k \cdot n \Delta(x, y) + k (\Delta_0 - \Delta(x, y))
\]

\[
e(x, y) = \exp \left[ i k \Delta_0 \right] \exp \left[ i k (n-1) \Delta(x, y) \right]
\]

\[
\begin{pmatrix}
U_{12} \\
U_{21}
\end{pmatrix}
= 
\begin{pmatrix}
U_{11} & U_{12} e^{i \Phi(x, y)} \\
U_{21} & U_{22}
\end{pmatrix}
\]
**Sign Convention**

\[ R > 0 \quad \text{concave surface} \]
\[ R < 0 \quad \text{convex surface} \]

\[ R_1 = \sqrt{R_1^2 - x^2 - y^2} \]

\[ \Delta_01 \]

\[ \Delta_1(x,y) = \Delta_01 - \left[ R_1 - \sqrt{R_1^2 - x^2 - y^2} \right] = \Delta_01 - R_01 \left[ 1 - \sqrt{1 - \frac{x^2 + y^2}{R_1^2}} \right] \]

Similar for 2nd half.

\[ R_2 = \sqrt{R_2^2 - x^2 - y^2} \]

\[ \Delta_03 \]

\[ \Delta_3(x,y) = \Delta_03 + R_2 \left[ 1 - \sqrt{1 - \frac{x^2 + y^2}{R_2^2}} \right] \]
Finally
\[ \Delta(x,y) = \Delta_{0} = R_{1} \left[ 1 - \sqrt{1 - \frac{x^{2} + y^{2}}{R_{1}^{2}}} \right] + R_{2} \left[ 1 - \sqrt{1 - \frac{x^{2} + y^{2}}{R_{2}^{2}}} \right] \]

\[ \Delta_{0} = \Delta_{01} + \Delta_{02} + \Delta_{03} \]

Paraxial approximation
\[ \Delta(x,y) = \Delta_{0} - R_{1} \left[ \frac{1}{2} \right] \left[ x^{2} + y^{2} \right] + R_{2} \left[ \frac{1}{2} \right] \left[ x^{2} + y^{2} \right] \]

\[ = \Delta_{0} - \left( \frac{x^{2} + y^{2}}{2} \right) \left( \frac{1}{R_{1}} - \frac{1}{R_{2}} \right) \]

and the phase function of the lens:
\[ t_{l}(x,y) = \exp[ik\Delta_{0}] \exp[-ik(n-1) \frac{x^{2} + y^{2}}{2f} \left( \frac{1}{R_{1}} - \frac{1}{R_{2}} \right)] \]

Lensesmakers formula:
\[ \frac{1}{f} = (n-1) \left( \frac{1}{R_{1}} - \frac{1}{R_{2}} \right) \]

and neglecting constant phase:
\[ t_{l}(x,y) = \exp[-ik \frac{x^{2} + y^{2}}{2f}] \]

This good for various lenses:

\[ \text{Double convex} \quad \text{Plano convex} \quad \text{Positive meniscus} \]

\[ \text{Plano concave} \quad \text{Convex concave} \quad \text{Negative meniscus} \]
Physical meaning

Converging!
Sph. wave
\[ e^{-j \frac{k}{2a}(x^2+y^2)} \]

Focusing
\[ e^{j \frac{k}{2a}(x^2+y^2)} \]

\[ f(z) \]

Tells us where is the 3-function
in space!

Fourier Transforming Properties of
a lens.

We will be studying various configurations in the past
used transparencies for Sig. Proc.
for SAR. Then - SLM. Now comp.

3 config: input 1, input 2, input 3
\[ V(x, y) = A \cdot t_a(x, y) \]

Aperture \( P(x, y) = \begin{cases} 1 & \text{inside aperture of lens} \\ 0 & \text{out.} \end{cases} \)

Then
\[ V'(x, y) = A \cdot t_a(x, y) \cdot P(x, y) e^{-i \frac{k}{2f} (x^2 + y^2)} \]

Then use FSP over \( z = f \).

\[ U_f(u, v) = \frac{\exp \left[ -i \frac{k}{2f} (u^2 + v^2) \right]}{j \cdot f} \frac{1}{2\pi} \int_{-\infty}^{\infty} V'(x, y) \exp \left[ -i \frac{k}{2f} (x^2 + y^2) \right] dx dy \]

and after sub of \( V' \)

\[ \mathcal{F}(u, v) = \frac{\exp \left[ i \frac{k}{2f} (u^2 + v^2) \right]}{j \cdot f} \frac{1}{2\pi} \int_{-\infty}^{\infty} V(x, y) \cdot P(x, y) \exp \left[ -i \frac{2\pi}{2f} (u x + v y) \right] dx dy \]

2-D FT.

If \( t_a(x, y) \) is small compared to \( P(x, y) \)

\[ U_f(u, v) = \frac{1}{j \cdot f} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} V(x, y) \exp \left[ -i \frac{2\pi}{2f} (u x + v y) \right] dx \right] \exp \left[ -i \frac{k}{2f} (u^2 + v^2) \right] dv \]

\[ f_x = \frac{u}{2\pi} \quad f_y = \frac{v}{2\pi} \quad \text{From holography} \]
FT of field has $Q$-phase

Intensity is OK:

$$I_d(u,v) = \frac{A^2}{A^2 + 2} \left| \mathcal{F}\{t_a(x,y)\} \right|^2$$

2) input $t_a(x,y)$ in front of the lens

\[
t_a(x,y)
\]

let $F_o(f_x, f_y) = \mathcal{F}\{t_a(x,y)\}$

$F_e(f_x, f_y) = \mathcal{F}\{U_e(x,y)\}$

Then $F_e(f_x, f_y) = F_o(f_x, f_y) \cdot \exp\left[-j\pi d (f_x^2 + f_y^2)\right]$ within a constant.

let $P(x,y)$ be 1

Then

\[
U_f(u,v) = \exp\left[\frac{jk}{2\pi} (u^2 + v^2)\right] \cdot \mathcal{F}\{U_e(x,y)\}
\]

and

\[
U_f(u,v) = \frac{\exp\left[\frac{jk}{2\pi} (u^2 + v^2)\right]}{j\pi f} F_o(f_x, f_y) \exp\left[-j\pi d (f_x^2 + f_y^2)\right]
\]

using

\[
\frac{2\pi}{\lambda} \cdot \int \frac{1}{2\pi} x^2 dx^2 = \frac{k d u^2}{2\pi f^2}
\]
we can rewrite

\[ U_f(x, y) = \exp \left[ i \frac{k}{2f} \left( 1 - \frac{d}{f} \right) \left( x^2 + y^2 \right) \right] \]

\[ \frac{j \pi}{\sin \theta} \int \frac{d \xi \eta}{f} \Phi \left( \frac{x}{\xi}, \frac{y}{\eta} \right) \]

Here we have a choice \( d = f \) such that we get exact \( F \) 1

Next \( \Rightarrow \) finite aperture of the lens \( \Rightarrow \) vignetting of input.

tan \( \theta = \frac{\xi}{\eta} \)

Effective aperture in beam optics approx.

Shift P \( (\xi - \frac{d}{f} x, \eta - \frac{d}{f} y) \) such that

\[ U_f(x, y) = \exp \left[ i \frac{k}{2f} \left( 1 - \frac{d}{f} \right) (x^2 + y^2) \right] \int \frac{d \xi \eta}{\sin \theta} \Phi \left( \frac{x}{\xi}, \frac{y}{\eta} \right) \]

\[ d \xi d \eta \]

\[ P \left( \xi + \frac{d}{f} x, \eta + \frac{d}{f} y \right) \exp \left[ -i \frac{2 \pi}{\lambda f} (x \xi + y \eta) \right] d \xi d \eta \]
3). Input behind the lens:

Using paraxial approx:

\[
U_0(x,y) = A \frac{1}{d} P \left( \frac{tx}{2d}, \frac{ty}{2d} \right) \exp \left[ -j \frac{k}{2d} (x^2 + y^2) \right]
\]

Then FSD we get \( z = d \):

\[
U_d(u,v) = \frac{1}{d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t_a(x,y) P \left( \frac{tx}{2d}, \frac{ty}{2d} \right) \exp \left[ -j \frac{2\pi}{d} (ux + vy) \right] dx dy
\]

Scaling of FT as we change \( d \)! Good for spot filtering to match scales!