Problem 1: Mirrors  
10 points  
To find minimum height of planar mirror for a 2m tall person to see themselves head to toe.

Method 1- Can the light from the toe incident on the planar mirror be reflected and received by the head(eyes)?

For the light path shown, $h_1 = h_2 = h_0$ and $\theta_1 = \theta_2$

![Figure 1: Problem 1a](image1)

Method 2- For a person to see the image of his/her toe, the boundary of the mirror limits the vision.

Thus in each case, $h_0 = 1m$

![Figure 2: Problem 1b](image2)

This does not depend on the distance of the person from the mirror!
Problem 2: Fermat’s Principle and Snell’s Law

To start with the Fermat’s Principle and derive the Snell’s Law.

To use Fermat’s principle we want to express the optical path (OP) in terms of a single independent variable x, and set $\frac{dOP}{dx} = 0$.

Define $x \triangleq$ height of intercept point B, as shown in fig??

\[
l_1 = \sqrt{d_1^2 + x^2} \quad l_2 = \sqrt{d_2^2 + (d - x)^2} \tag{2}
\]

\[
OP = n_1 l_1 + n_2 l_2 = n_1 \sqrt{d_1^2 + x^2} + n_2 \sqrt{d_2^2 + (d - x)^2} \tag{3}
\]

\[
\frac{dOP}{dx} = \frac{d}{dx} n_1 \sqrt{d_1^2 + x^2} + \frac{d}{dx} n_2 \sqrt{d_2^2 + (d - x)^2}
\]

\[
= n_1 (d_1^2 + x^2)^{-\frac{1}{2}} \cdot 2x + n_2 (d_2^2 + (d - x)^2)^{-\frac{1}{2}} \cdot 2(d - x) \cdot (-1) = 0 \tag{4}
\]

\[
\Rightarrow \quad n_1 \frac{x}{l_1} = n_2 \frac{d - x}{l_2} \tag{5}
\]

from the diagram we have $\sin \theta_1 = \frac{x}{l_1}$, $\sin \theta_2 = \frac{d - x}{l_2}$. So $n_1 \sin \theta_1 = n_2 \sin \theta_2$

![Figure 3: Problem 2](image-url)
Problem 3: Ideal Prism

To understand refraction through an ideal prism, and characterize the given $\theta$.

Applying Snell’s Law to the prism-air interface,

$$\eta_{\text{prism}} \sin \theta_0 = \sin \theta_1$$

Using geometry, we can deduce that $\theta_0 = 10^\circ$, thus given $\theta = \theta_1 - \theta_0$

![Diagram of light passing through a prism]

Figure 4: Problem 3

Then, substitution and plotting in MATLAB gives us:

![Graph showing transmitted light angle vs index n]

Figure 5: MATLAB Result

Then, maximum $\theta = 80^\circ$ and $\eta_{\text{prism}} = 5.7588$

Note: the minimum deflection angle here is zero, at $\eta_{\text{prism}} = 1$ (i.e., when there is no prism at all)
Problem 4: Surface Reflections  
6 points  
To find the % surface reflections incident on a surface of diamond submerged in water.

Values of $\eta_{\text{diamond}} = 2.419$ and $\eta_{\text{water}} = 1.333$  
Formula for reflected intensity,

$$\left(\frac{2.419 - 1.333}{2.419 + 1.333}\right)^2 = 8.378\%$$

Note: For refractive indices, any reasonable value is allowed.

Problem 5: Fluorescent Slabs  
8 points  
To find the total fraction of light intensity that escapes the top surface of silicon into air.

The light energy (or intensity) that escapes from the top surface is the light that is not totally internally reflected. That is,

![Figure 6: Visualizing Problem 5](image)

Thus $\theta_c$ at the silicon-air interface can be found using Snell's Law to be $16.6015^\circ$  
Solid angle is then, $2\pi (1 - \cos \theta_c)$

Total fraction $= \frac{2\pi (1 - \cos \theta_c)}{4\pi} = 2.08\%$

While the method described above is a decent approximation, this problem requires a more involved thought process, in that light is isotropically radiated out the top and bottom faces of the slab. One must note that at each surface, light rays are also subject to Total Internal Reflection, resulting in what may seem light a geometric progression of reflected and transmitted rays bouncing off the surfaces. What could make the final solution easier, then, is the underlying symmetry between the top and bottom cones of light emitted. So one could say, the fraction of the light that comes out the top surface is half the total fraction of light, computed using the solid angles and gives the same result.

![Diagram of light reflection and transmission](image)
Problem 6: Slab Waveguides

To determine optimal conditions on $n_{cladding}$ in order to achieve a waveguiding structure (TIR) for light propagation.

(A):
Applying Snell’s Law at 45° incidence (air-core interface), $n_{air} \sin \theta_1 = n_{core} \sin \theta_2$.
Since $n_{core} = 1.5$, $\theta_2 = 28.13^\circ$

Thus $\theta_3 = \pi/2 - \theta_2 = 61.87^\circ$ at the core-cladding interface
For a waveguiding structure, the cladding must completely contain the light inside the core and total internal reflection must occur.

$n_{core} \sin \theta_3 = n_{cladding}$ implies max $n_{cladding} = 1.323$ for $\theta_c$

(B):
The tapered structure is insufficient to contain the incident light inside the core, and hence achieve TIR as the incidence angle ($\theta_3'$) at the core-cladding interface decreases, thereby also reducing $n_{cladding}$. This way, the light will not be able to propagate to Face 2.

$n_{cladding} = n_{core} \theta_3'$.

Also note here that as light propagates through the waveguide, with the two surfaces not being parallel anymore, the incidence angle at the cladding for the second TIR has $\theta_1 < \theta_0$ with the next incident angle $\theta_2$ being smaller (see figure below). We can see that the incidence angle at the cladding goes on decreasing, making it hard for us to ensure TIR each time for light propagation.

For incidence on Face 2, the incident angle ($\theta_1''$) at the core-cladding interface is now larger, making the minimum index condition for the cladding sufficient as calculated in (A)

(Note: Do make sure all diagrams are neat, since it’s important to know what angles you’re looking at!)
Problem 7: Retroreflectors  
30 points
To deduce refractive index considerations for a retroreflector at different incident angles.

(A):
For normal incidence, we know that light passes through the front face without bending. This light ray is then incident on the rear surface at a 45° angle.

Applying TIR condition at glass-air interface,

\[ \eta_{\text{solid}} \sin(45°) = 1 \sin(90°) \]

Or, \( \eta_{\text{solid-min}} = 1.414 \)

Note that we get retroreflection for all normally incident light, which rules out the need for the rear surfaces to be mirror-coated.

(B):
Remember, our goal is to get some retroreflection from any angle between 0° and 90°. Let’s consider the following cases for various incidence angles:

Case 1: For \( \theta_i = 0° \), as in (A), we know we get retroreflection due to TIR at the two rear surfaces.

Case 2: As incidence angle on the front face increases, we cannot get retroreflection from the upper parts of the solid. However, the lower parts of the solid, at the same incidence angle, give us retroreflection as shown in the following figure.
Case 3: (to be dealt with caution) Now, our extreme condition is $\theta_1 = 90^\circ$, also known as the grazing angle. From the figure that follows, we discuss two sub-cases, i.e. one in which the ray transmitted from the front facet prescribes a $\theta_1 > 45^\circ$, and another in which this ray prescribes $\theta_1 \leq 45^\circ$.

In the first sub-case, we notice that we don’t get retroreflection ever, while in the second, we do. From here we conclude that the transmitted angle must be less than or equal to $45^\circ$ for TIR on both the rear surfaces. Applying this condition on the front facet then,

$$\eta_{\text{solid}} \sin(\theta_t) = 1 \sin(90^\circ)$$

Or, $\eta_{\text{solid}} \geq 1 / \sin 45^\circ \geq 1.414$

Sub-case 1

Sub-case 2

To be sure our light is indeed undergoing TIR at both the rear surfaces, we must satisfy both the grazing angle and the TIR conditions at the front and rear surfaces respectively. In order to do this,

We have $\theta_t + \theta_{\text{int}} = 45^\circ$

Grazing incidence condition: $1 \sin(90^\circ) = \eta_{\text{solid}} \sin(\theta_t)$

TIR condition at rear surface: $\eta_{\text{solid}} \sin(\theta_{\text{int}}) = 1 \sin(90^\circ)$

Substituting, we get $\theta_t = \theta_{\text{int}} = 22.5^\circ$

Thus, $\eta_{\text{solid}} \geq 1 / \sin(22.5^\circ) = 2.61$
(C):
The treatment of this question is very similar to (B), thus the minimum index is still 2.61. But we notice that there is some case for which we can never get retroreflection. Consider the following case of grazing angle incidence:

If we combine the cases of incidence at upper parts of the solid and grazing angle, or in other words, if we have a grazing light ray incident at the upper part of the solid, only if \( \theta_i \) at the rear surface is 45\(^\circ\) can the reflected light transmitted along the left boundary be retroreflected. This means that the ray in the figure is normal to the left boundary, and Snell’s Law gives us,

\[
\eta_{\text{solid}} \sin(0^\circ) = 1 \cdot \sin(90^\circ)
\]

Or, \( \eta_{\text{solid}} = \infty \), which is impossible!

(D):
This could be tricky. While metals like Gold do have high reflectivities, the refractive index also consists of imaginary parts, that make it absorb a portion of the light. Since TIR corresponds to perfect reflection, using Gold coating will reduce the light intensity of the reflected light.