Homework 5 Solutions

The solution to "Problem 0" (the thin prism) is on the last page

Problem 1: The grating equation
To find the maximum diffraction orders of a grating of given period using the grating equation.

Given, $\lambda = 1094$ nm, $\theta_i = 0$, $d = 7$ $\mu$m
We know that the grating equation is given by,

$$m\lambda = d(sin\theta_i + sin\theta_r)$$

Where $m$ can be + or - orders of the diffraction grating, $\lambda$ is the incident light wavelength, $\theta_i$ is the incident angle and it is important to note that $\theta_r$, the angle the orders are diffracted at, is taken as negative, defined by the convention (taken from left to right from the normal, see figure 1).

(1): We can find the maximum orders than can be obtained from this grating, given a period of 7 $\mu$m by noticing that the number of orders will be diffracted at an angle less than 90°. Thus the maximum order $M_{max} = \pm 6$.

(2): The energy distribution of the diffracted orders depends on a number of parameters, like the power and polarization of the incident light, the angles of incidence and diffraction, the (complex) index of refraction of the materials at the surface of the grating, and the period of the grating itself. (An interesting read would be something called a blaze condition for reflection gratings).

For the given period, just by observing the grating equation, we can say that a shorter $\lambda$ would give us higher orders. Additionally, we have assumed normal incidence of light, so by observing a tilt in the incident light ray, we may obtain higher orders (since $\theta_i$ increases from 0 to 1). Let’s take to Matlab to see what happens when we consider changing incident angles (Figure 2).
(3):
Let $\lambda_r = 632$ nm and $\lambda_b = 474$ nm. Then from the grating equation, when $\theta_i = 0$:

$$M\lambda_r = d\sin\theta_r$$
$$N\lambda_b = d\sin\theta_r$$

Thus,

$$\frac{M}{N} = \frac{\lambda_b}{\lambda_r} = \frac{474\text{nm}}{632\text{nm}} = \frac{3}{4}$$

Figure 1: Monochromatic beam incident on (blazed) diffraction grating at angle $\theta_i$ and diffracted at angle $-\theta_r$. The blaze spacing is $d$.

Figure 2: Observing the effect of tilt
Problem 2: Far field from a square aperture

To calculate the far-field image intensity distribution for an object transparency illuminated with collimated 0.55 micron light.

(1):

Using the Antenna Designer’s Formula, we can say that the far-field condition (or the Fraunhofer condition) is that,

\[ d > \frac{2D^2}{\lambda} \]

Thus the distance \( d \) we must go to obtain our far-field diffraction pattern should be greater than 363.63 meters.

However, one must carefully note that in Optics, we impose a more stringent condition by means of the Fraunhofer Approximation. We say that our Fresnel Number for far-field should be much lesser than 1. That is, \( N_F \ll 1 \) or,

\[ \frac{D^2}{\lambda d} \ll 1 \]

Here \( d \) is the distance from the aperture while \( D \) is the largest aperture size. The way to go about this would be to choose an appropriate Fresnel number and find \( d \).

(2):

Now, if we go 10 kms away, we could derive our intensity distribution as follows:

First is to acknowledge that 10 kms definitely lies in the Fraunhofer region, and hence the intensity distribution will just be an amplitude square of the Fourier Transform of the aperture. Let the aperture transmission amplitude be, in the case of a box-like rectangle:

\[ t(x, y) = \text{rect}(\frac{x}{w_x})\text{rect}(\frac{y}{w_y}) \]

where \( x \) and \( y \) are the widths of the aperture in \( x \) and \( y \) directions respectively. The fourier transform of this is just:

\[ w_xw_y\text{sinc}(w_xf_x)\text{sinc}(w_yf_y) \]

evaluated at \( f_x = \frac{x}{\lambda z}, f_y = \frac{y}{\lambda z} \), or:

\[ w_xw_y\text{sinc}(\frac{w_xx}{\lambda z})\text{sinc}(\frac{w_yy}{\lambda z}) \]

The diffraction pattern will also have the additional phase terms as:

\[ \frac{e^{jkz}e^{jk(x^2+y^2)/2z}}{j\lambda z} \]
However, when computing the intensity, we will simply obtain:

\[ I = \frac{w_x^2 w_y^2}{\lambda^2 z^2} \text{sinc}^2 \left( \frac{w_x x}{\lambda z} \right) \text{sinc}^2 \left( \frac{w_y y}{\lambda z} \right) \]

Substitute the values of the aperture widths and the distance \( z = 10 \text{ kms} \) in the above equation and simplify the intensity pattern.

**Problem 3: Diffraction from a circular aperture**

To understand the effects of diffraction by calculating the spot size of the pattern produced through a circular aperture on the moon.

(A): We know that, the diffraction pattern from a circular aperture is also known as the Airy Pattern, for which the width of the central lobe is given as:

\[ d = 1.22 \frac{\lambda z}{D} \]

where \( \lambda = 632.8 \text{nm} \) for a typical He-Ne red laser, \( D \) is the aperture diameter and \( z \) is the distance to the moon, roughly 384400 km. Substituting these values, we get \( d \), or spot size as **29.7 km**.

(B): Intuitively, to reduce the spot size on the moon from 29.7 km to barely 1 mm, we’ll need a very large aperture \( D \), looking at the formula. Thus,

\[ D = \frac{1.22 \lambda z}{d} \]

where \( d = 1 \text{ mm} \). We get \( D \) as **594 km**. To be able to collect all the light from such a large aperture and focus it into such a spot, we will need a converging lens of focal length roughly equal to \( z \) (distance to the moon). The \( F/\# \) (focal length/aperture) is then just \( z/D = 384400/594 = 647.138 \) which is **\( F/\#647 \)**!
Problem 4: Imaging system resolution
To study the effects of diffraction that limits the resolution of an imaging system, e.g. the Hubble Telescope of $D = 2.4\text{ m}$, $f = 57.6\text{ m}$.

(1):
Assuming we are looking at a distant star emitting $\lambda = 500\text{ nm}$, minimum angular resolution is simply,

$$\theta = \frac{\lambda}{D} = 2.54 \times 10^{-7}\text{ rad}$$

(2):
The diameter of the spot is $2\theta f = 29.3\ \mu\text{m}$. The intensity distribution is:

Note that in the figure, we have the central lobe width as $1.446 - (-1.446) = 2.892 \times 10^{-5}$ which matches our calculated diameter spot!

(3):
If the Hubble telescope landed at UCSD to observe distant stars and other astronomical objects, the imaging would suffer from various physical effects of the atmosphere majorly like pollution, stray light, wind, temperature and varying refractive index between the celestial body and the telescope lens (similar to the twinkling effect of stars!) due to its large aperture. How to resolve this? Adaptive Optics. We could add a dynamic phase element and control it in real time such that any wavefront deformations from a distant object can be minimized.

Read: http://www.skyandtelescope.com/astronomy-news/next-gen-adaptive-optics-09092014/
Problem 5: Magnifying "glass"

To see how a concave reflector of radius \( r = 10 \text{ cm} \) and aperture diameter \( D = 1 \text{ cm} \) held in a glass cup acts as a magnifying lens.

(1):
The focused height for a concave mirror is half its radius of curvature, i.e. \( H = r/2 = 5 \text{ cm} \). The sample must be placed here to obtain an image at infinity.

(2):
The diffraction limited spatial resolution when wavelength \( \lambda = 550 \text{ nm} \) is traveling in a medium with refractive index \( n \) and converging to a spot with half-angle \( \theta \) can be calculated as (note: read on Abbe's Diffraction Limit):
The half angle converges the spot into a radius,

\[
d = \frac{1.22 \lambda}{2n \sin \theta}
\]

where, \( \theta \) can be obtained as (see figure):

\[
\theta = \tan^{-1}\left(\frac{D/2}{H}\right) = 0.0997
\]

And thus, \( d = 3.37 \mu m \); note that \( n \sin \theta \) is also known as the Numerical Aperture (N.A.) of the reflector, which is its light gathering ability.

(3):
When we have a liquid of refractive index \( n = 1.33 \) in the cup surrounding the reflector:

- the focal length or focused height \( (H) \) does not change as the sample will still have to be placed at this distance for the image to be formed at infinity.

- the diffraction limit changes, however, as it is a function of refractive index: \( d = 2.0773 \mu m \). Note that the diffraction limit has decreased, thus providing a higher resolution image of the sample species.
**Problem 4: Thin (paraxial) Prism**

To compute the complex amplitude transmittance of a thin prism with a small angle $\alpha$ and a maximum thickness $d$.

From (2.4-4), Saleh & Teich, ‘Fundamentals of Photonics’ (2nd Edition), we know that the transmittance of a variable thickness plate is:

$$t(x, y) = h_0 e^{-j(n-1)k_0 d(x,y)}$$

where $d(x, y)$ is the smoothly varying thickness in $x$ and $y$ and $h_0 = e^{-j k_0 d_0}$ is a constant phase factor associated with the maximum thickness $d_0$.

![Diagram](image)

**Figure 2.4-6** Transmission of a plane wave through a thin prism.

Since angle $\alpha$ is small, we have,

$$\tan(\alpha) = \frac{d(x, y)}{x} \Rightarrow \alpha \approx \frac{d(x, y)}{x} \Rightarrow d = \alpha x$$

Substituting in transmittance function,

$$t(x, y) = h_0 e^{-j(n-1)k_0 \alpha x}$$

(Alternatively, notice that the phase shift induced by the prism is, $e^{-j(k_0 n)d(x,y)}$ and the phase shift introduced by the air is $e^{-j k_0 d_0(x,y) - d_0}$, giving us a net phase of $e^{-j k_0 d_0} e^{-j k_0 (n-1) d(x,y)}$.)